Harmonic Tide Solutions

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1 Harmonics

Let a tide process be denoted by

$$\mathcal{T} = (\mathbf{M}, \zeta)$$

where M is the vertically integrated current vector (areal flow density or specific mass transport vector) and ζ the surface elevation. They are functions of position and time

$$\mathcal{T} = \mathcal{Z}(x, y, t)$$

The finite difference solution presents elevations and currents on two staggered grids in space and at discrete steps in time.

We will stick to describing a harmonic solution of ζ. The currents are described equivalently.

There might be a large number of tidal harmonics be represented in the excitation of the model (boundary conditions, external tidal forces). If everything were linear, the theorem of linear superposition would hold. If the forcing potential for instance is

$$\phi(x, y, t) = \sum_k \Phi_k(x, y) e^{i\omega_k t}$$

and the solution

$$\zeta(x, y, t) = \sum_k Z_k(x, y) e^{i\omega_k t}$$

we could compute a partial solution for each k,

$$\zeta_k(x, y, t) = Z_k(x, y) e^{i\omega_k t}$$

by exciting the model with only one harmonic

$$\phi_k(x, y, t) = \Phi_k(x, y) e^{i\omega_k t}$$

The complex field quantities $Z_k(x, y)$ and $\Phi_k(x, y)$ are instances of harmonic representations of an oscillating process. They can be represented by either their real and imaginary parts $(\mathcal{R}e, \mathcal{I}m)$ or by amplitude and phase (A, δ)

$$\begin{aligned} A &= \sqrt{\mathcal{R}e^2 + \mathcal{I}m^2} \\ \delta &= \arctan \frac{\mathcal{I}m}{\mathcal{R}e} \end{aligned}$$

1.1 Amphidromes

The particle trajectory for a small tide (elevation negligible with respect to depth) in a circular, flat-bottomed basin on a rotating earth (i.e. with Coriolis accelerations present) during one tidal cycle is a small circle; all particles move in parallel. The continuity equation will cause a surface elevation in the form of a uniform slope with zero elevation in the centre. In the centre the elevation will be zero. Imagine the gradient of the surface pointing from the highest point to the lowest. This gradient is a straight line and the time delay of the elevation maximum with respect to the maximum of the excitation force divided by the duration of a full cycle is a measure for the phase of the oscillation.

Hence we can draw lines along the basin of equal phase, when the maximum will be reached, and we relate this phase to the maximum of the excitation force at an agreed-upon point on the earth (the astronomical tide at Greenwich meridian for instance). We can accompany the phase lines by those that denote lines of constant amplitude. By this we create a spider-web pattern, with the phase lines like rays outward from the centre and the amplitude lines as concentric circles. This is the generic case of an *amphidrome*.

An *amphidromic point* (the centre of the amphidrome, where amplitude is zero) marks the centre of an oscillating subsystem. In a wide and shallow basin with arbitrary land boundaries several amphidromes will be present. They line up offshore such that there is a continuosly progressive wave along the coast (so-called Kelvin wave). Because of currents not being able to cross land boundaries, phase lines will tend to assume a right-angle orientation with respect to the coast. At capes the resulting pattern is as if there would be an amphidrome on land. This situation is sometimes called *virtual amphidrome*, perhaps reminding of the vritual electric charges used in electrostatics to draw field lines. Also, there is a general tendency of amplitude lines intersecting phase lines at right angles, a consequence of the wave equation, the exactness being challenged by bottom topography and complicated land boundaries.

At the equator the Coriolis forces become zero and the tendency of oscillating systems to form amphidromes is very low.

1.2 How harmonic tide solutions are computed in OTEQ/TTEQ

We mentioned above the principle of linear harmonic superposition. In the simple case of one-frequency excitation, the solution will contain, in addition to the predominant harmonic solution, some transient behaviour from the stage when we switched the model on, and there can be some numerical noises (round-off errors) ϵ

The sum

$$Z_k^{(N)}(x, y) + E_k^{(N)} = \frac{2}{N} \sum_{n=0}^{N-1} (\zeta(x, y, n\Delta t) + \epsilon_n) e^{-i\omega_k n\Delta t}$$

will (hopefully) converge to a constant nonzero Z and a zero error E as $N \rightarrow \inf$. This equation can be looked upon also as a frequency selective filter.

In a more realistic case involving nonlinearity (shallow water, nonlinear bottom friction, advection), linear superposition will not hold. The model must therefore be excited with a wide spectrum of tides simultaneously. Still we might like to extract the harmonic constituents of the solution. The nonliear processes will create intermodulation products like upper harmonics (at double, triple etc. frequencies).

Since N will be finite, the capability of the sum to extract the result for one frequency while $\zeta(x, y, t)$ is rich of frequencies is limited. It is largely described by the Fourier theorem concerning the antagonism between frequency resolution and time limitation.

The *relative* error when extracting a large harmonic process out of a pool of precesses with similar frequencies is smaller than extracting a small one. Thus, it is only realistic to solve for dominating tides, the largest in each band (diurnal, semidiurnal) and the intermodulation by-products in the high-frequency bands (terdiurnal, quaterdiurnal, etc.). The minimum requirement to discern solar from lunar tides is a time series of half a month, yet one month gives substantially better results (since the half-month suggestion would ignore the presence of other tides near M_2 and S_2.

So, finally, this is what is returned by subroutines OTEQ and TTEQ in array `ZSUM` and put on disk by program otemt1 through file unit 41:

$$Z_r = \sum_{n=N_0}^{N-1} \zeta_{ij}^{(n)} e^{-i\omega_k n\Delta t}$$

Z_r is a *packed array* that avoids to store land points. Its Fortran precision is `COMPLEX*16`, therefore it's huge anyway. Thus, there is a unique mapping between array index r and location (i, j).

Next to `ZSUM` in the subroutine parameter list is `NRSUM`. It returns the value of $N - N_0 + 1$, so that `NRSUM/2` must be used to normalize `ZRSUM`. The harmonic array is saved unnormalized because we want to be able to continue to accumulate `ZRSUM` in successive calls of the tide equation solver.